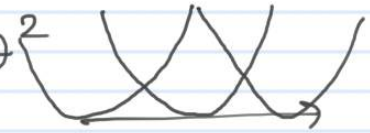
つくば大 集中講義 Recent studies on gauge theories using 't Hooft anomalies. 谷崎 佑弥 (YITP) 2020.12

§3. Yang-Mills theory at $\theta=\pi$

Outline

- Yang-Mills theory : θ-dynamics $\Rightarrow E_{G.S.}(\theta) = \frac{\chi_{top}}{2}\theta^2$



$$E_{G.S.}(\theta) = \min_{k\in\mathbb{Z}} \frac{\chi_{top}}{2}(\theta-2\pi k)^2.$$

$\rightsquigarrow$ reinterpretation as Witten effect of monopole

- Center sym. $\Rightarrow$ $\mathbb{Z}_N^{(1)}$ symmetry (1-form)
- $B \longleftrightarrow$ Backgrd of $\mathbb{Z}_N^{(1)}$-sym. $\rightsquigarrow$ $\mathbb{Z}_N^{(1)}\times CP$ mixed 't Hooft anomaly

$\Rightarrow \begin{cases} \cancel{CP}, \text{ or} \\ \cancel{\mathbb{Z}_N^{(1)}} \end{cases}$

§3.1 θ-dynamics

Notation : $a = a_\mu^a T^a dx^\mu$: $SU(N)$ gauge field.

(a: SU(N)-label; μ: Lorentz index; T^a: Hermitian)

$\Rightarrow$ Field strength $f = da + i\,a\wedge a = \frac{1}{2}\underbrace{(\partial_\mu a_\nu - \partial_\nu a_\mu + i[a_\mu, a_\nu])}_{f_{\mu\nu}} dx^\mu\wedge dx^\nu$

(covariant derivative $D_\mu = \partial_\mu + i a_\mu$)

$$S_{YM} = \frac{1}{g^2}\int \mathrm{tr}(f\wedge *f)$$

(g^2: coupling.)

YM theory with the θ-angle.

$$S = \frac{1}{g^2}\int \mathrm{tr}(f\wedge *f) + i\theta \underbrace{\frac{1}{8\pi^2}\int \mathrm{tr}(f\wedge f)}_{Q_{top.}\in\mathbb{Z}}.$$

(We only consider 4-dim spin mfd.)

(Index theorem : We can consider fund. Dirac fermion. $\not{D}$

$$\Rightarrow \mathbb{Z} \ni \mathrm{Index}(\not{D}) = \underset{\text{dim}}{\ker(\not{D})} - \underset{\text{dim.}}{\mathrm{coker}(\not{D})} = \frac{1}{8\pi^2}\int \mathrm{tr}(f\wedge f) \quad)$$

We define the partition function

$$Z(\theta) = \int \mathcal{D}a \, \exp\left(-\frac{1}{g^2}\int \mathrm{tr}(f\wedge *f) - i\frac{\theta}{8\pi^2}\underbrace{\int \mathrm{tr}(f\wedge f)}_{\in\mathbb{Z}}\right)$$

$$\| $$

$$e^{-\mathrm{Vol}_4 \cdot E_{G.S.}(\theta)}$$

$$\rightarrow \boxed{E_{G.S.}(\theta+2\pi) = E_{G.S.}(\theta).}$$

- Large-N limit $\frac{1}{\hbar}$ $\quad N\to\infty$, $\quad \lambda = g^2 N$: fixed.

$$Z(\theta) = \int \mathcal{D}a \, \exp\left(-N\left[\frac{1}{\lambda}\int \mathrm{tr}(f\wedge *f) + i\frac{\boxed{\theta(N)}}{8\pi^2}\int \mathrm{tr}(f\wedge f)\right]\right)$$

$$\xrightarrow{N\to\infty} e^{-\mathrm{Vol}_4 \cdot E_{G.S.}(\theta)}$$

How does it look like in the N-limit?

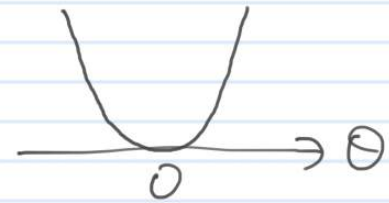
$$E_{G.S.} = N^2 \cdot f\left(\underbrace{\frac{\theta}{N}}_{\text{small}}\right) = \underbrace{N^2 f(0)}_{\doteq 0} + \frac{f''(0)}{2}\theta^2 + \cancel{\frac{f^{(4)}(0)}{4!}\left(\frac{\theta^4}{N^2}\right) + \cdots}$$

$\theta \sim 2\pi$ ..

$f''(0) = \chi_{top}$

$\mathcal{O}\left(\frac{1}{N^2}\right)$

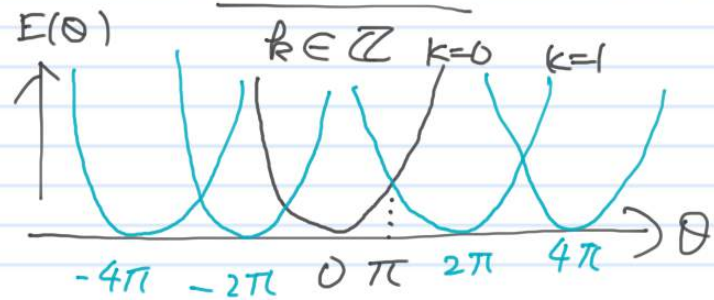
$$\boxed{E_{G.S.}(\theta) = \frac{\chi_{top}}{2}\theta^2.}$$



(?) $E_{G.S.}(\theta+2\pi) = E_{G.S.}(\theta)$.

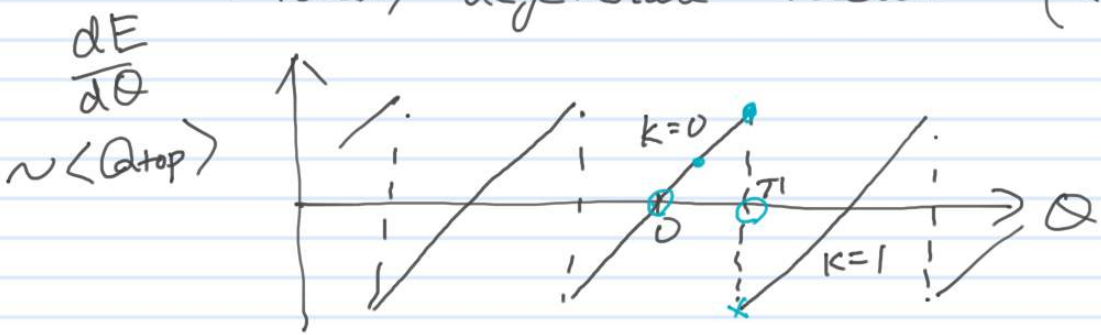
Witten '80, '98 $\Rightarrow$ To make these consistent, we must introduce the multi-branch structures

$$E_{G.S.}(\theta) = \min_{k\in\mathbb{Z}} \frac{\chi_{top}}{2}(\theta-2\pi k)^2.$$



$\Rightarrow$ at $\theta=\pi$, $k=0$ and $k=1$ states have the same energy.

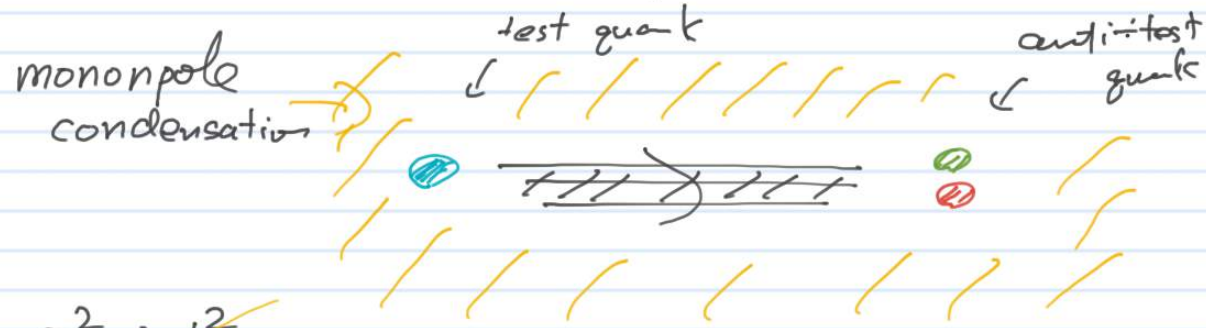
$\Rightarrow$ Doubly degenerate vacua ($\longleftrightarrow$ CP-symmetry breaking)



CP transformation.

$\hookrightarrow$ spontaneous CP breaking.

Reinterpretation of multi-branch str. from monopole condensation



ele/mag charge

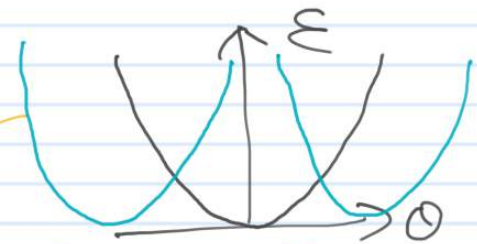
monopole: $(E, M) = (0, 1)$

$\downarrow$ Coulomb energy

$$\mathcal{E} \sim E^2 + M^2 = 0^2 + 1^2$$

$\Downarrow$ θ - angle

$$\mathcal{E} \sim \Big(E + \underbrace{\tfrac{\theta}{2\pi} M}_{\text{witten effect}}\Big)^2 + M^2 = \left(\tfrac{\theta}{2\pi}\right)^2 + 1^2$$



$\mathbb{Z}$

<u>dyon</u>: $(E, M) = (-k, 1)$

$k = -1$ $\quad$ $k = 1$

$\Downarrow$ Coulomb energy with θ

$$\mathcal{E} \sim \left(E + \tfrac{\theta}{2\pi} M\right)^2 + M^2 = \Big(\underbrace{-k + \tfrac{\theta}{2\pi}}\Big)^2 + 1$$

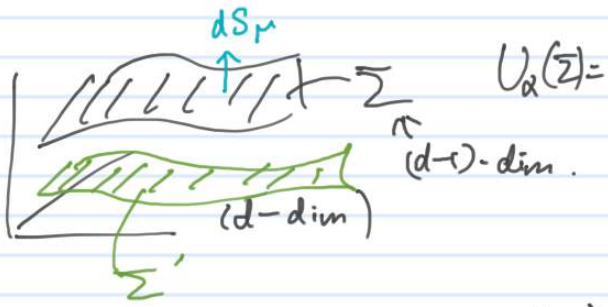
$\Rightarrow$ at $\theta = \pi$, mon. ($k=0$) condensation & dyon ($k=1$) condensation have the same energy $\rightsquigarrow$ Spontaneous CP-breaking. (Cardy-Rabinovici '82.)

- $\mathbb{Z}_N^{[1]}$ symmetry (1-form symmetry)

* Global symmetry (Gaiotto, Kapustin, Seiberg, Willet '14)

<u>example</u> $U(1)$ continuous symmetry $\rightsquigarrow$ j^μ : Noether current

(Up to eq. of motion) $\partial_\mu j^\mu = 0.$



$$U_\alpha(\Sigma) = \exp\left(i\alpha \int_\Sigma \underbrace{dS_\mu}_{(d-1)\text{-form.}} j^\mu\right) = \exp\left(i\alpha \int_{\Sigma'} dS_\mu j^\mu\right) = U_\alpha(\Sigma')$$

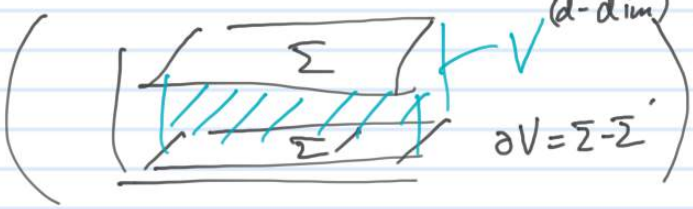
(proof)

$$U_\alpha(\Sigma)\cdot U_\alpha(\Sigma')^{-1} = \exp\left(i\alpha \int_{(\Sigma-\Sigma')} \overset{\overset{j}{\parallel}}{dS_\mu j^\mu}\right)$$

$$= \exp\left(i\alpha \int_{\partial V} j\right)$$

Stokes thm

$$= \exp\left(i\alpha \int_V \underbrace{dj}_{\partial_\mu j^\mu d^d x \,=\, 0}\right) = 1$$



$$\langle U_\alpha(\Sigma)\, \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle = \langle U_\alpha(\Sigma')\, \mathcal{O}(x_1)\cdots\mathcal{O}(x_n)\rangle$$

(as long as $\Sigma \longrightarrow \Sigma'$ does not cross $x_1, \cdots, x_n$)

Since we neglect the equation of motion

When $\Sigma \to \Sigma'$ crosses x_1, but does not cross $x_2, \cdots, x_n$,

$$\langle U_\alpha(\Sigma)\, \mathcal{O}(x_1) \cdots \mathcal{O}(x_n) \rangle = \langle U_\alpha(\Sigma') \left(e^{i\alpha Q} \mathcal{O}(x_1)\right) \mathcal{O}(x_2) \cdots \mathcal{O}(x_n) \rangle$$

$\leftarrow$ U(1) charge of $\mathcal{O}$

When we discuss symmetry

$\to \partial_\mu$ and we discuss $\underline{\partial_\mu(x)\, \mathcal{O}(y)}$ $\leftarrow$ OPE

$\Updownarrow$

We construct a "topological" operator $\underline{\underline{U(\Sigma)}}$, equivalent.

$$\underline{U(\Sigma)\, \mathcal{O}(x) = U(\Sigma')\, e^{i\alpha Q}\, \mathcal{O}(x)}$$

$\leftarrow$ usual symmetry

<u>Def</u> (<u>0-form</u> symmetry)

A QFT has a 0-form symmetry G,

$\overset{\text{def}}{\Longleftrightarrow}$

- Operators in each $(d-1)$-dim (closed) submanifolds M_{d-1},

$$U_g(M_{d-1}) \qquad (g \in G).$$

- $U_g(M_{d-1})$ is a topological operator ($\Longleftrightarrow$ Conservation law)
- $\forall g_1, g_2 \in G \qquad U_{g_1}(M_{d-1})\, U_{g_2}(M_{d-1}) = U_{g_1 g_2}(M_{d-1})$
- (Local operator) $\mathcal{O}(x)$ $\leftarrow$ (in a suitable basis)

$$U_g(S^{d-1}_x)\, \mathcal{O}(x) = R(g) \cdot \mathcal{O}(x)$$

$\leftarrow$ irreducible rep. R of G

//

<u>Def.</u> (p-form symmetry) (* If $p=0$, then this is an ordinary sym.)

A QFT has a p-form symmetry G

$\Longleftrightarrow$ For each codim. $(p+1)$ - submanifold $M_{(d-p-1)}$,

we have an operator $\underbrace{U_g(M_{(d-p-1)})}_{\text{charge operator}}$ $(g \in G)$.

They satisfy

- $U_g(M_{d-p-1})$ is a topological operator.
- $U_{g_1}(M_{d-p-1})\, U_{g_2}(M_{d-p-1}) = U_{g_1 g_2}(M_{d-p-1})$
- For (p)-dim operator $V(C_p)$, p-dim submanifold

$$U_g(S^{d-p-1}_{x \in C_p})\, V(C_p) = R(g) \cdot V(C_p)$$

they have a nontrivial linking number.

(Goal) We want to see that $SU(N)$ YM theory has $\mathbb{Z}_N^{(1)}$ symmetry.

Before that, let us check simpler examples.

<u>Example</u> 4d Maxwell theory.

a : $U(1)$ gauge field $\qquad S = \frac{1}{2g^2}\int da \wedge * da.$

(* This theory does not have much "ordinary" symmetry.
(It only has Poincaré, charge-conjugation))

- This theory has $U(1)^{(1)}_E \times U(1)^{(1)}_M$ symmetry.

Maxwell equation

$$d\Big(\underbrace{* \tfrac{1}{g^2} da}_{\leftarrow \text{2-form}}\Big) = 0 \qquad (\leftarrow \text{equation of motion})$$

$$d\Big(\underbrace{da}_{\text{2-form}}\Big) = 0 \qquad (\leftarrow \text{Bianchi identity}).$$

(* Ordinary $U(1)$ sym. : $\partial_\mu j^\mu = 0$.

$\iff j = \frac{1}{(d-1)!} j^\mu \varepsilon_{\mu\mu_1\cdots\mu_{d-1}} dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_{d-1}}$ satisfies $dj = 0$

$\iff \exp\left(i\alpha \int_{M_{d-1}} j\right)$ is a topological operator ($\leftarrow$ Stokes theorem))

Exactly in the same way, we have topological codim-2 operators.

$$\begin{cases} U^E_\alpha(M_2) = \exp\left(i\alpha \int_{M_2} \frac{1}{g^2} * da\right) \\ U^M_\beta(M_2) = \exp\left(i\beta \int_{M_2} \frac{1}{2\pi} da\right) \end{cases} \qquad \leftarrow U(1)\text{-symmetry}$$

Wilson loop & 't Hooft loop.

- $W(C) = \exp\left(i \int_C a\right)$ (C: 1-dim closed loop)
- $H(C)$ ← defect operator, s.t.

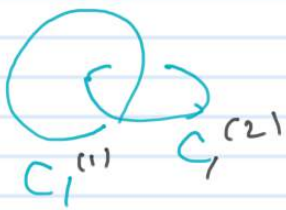
$$\int_{S^2} da = 2\pi \quad \text{if } S^2 \text{ wraps around } x \in C.$$

They satisfy

$$\begin{cases} U_\alpha^E(M_2)\, W(C) = e^{i\alpha \cdot \mathrm{Link}(M_2, C)}\, W(C) \\ U_\beta^M(M_2)\, H(C) = e^{i\beta \cdot \mathrm{Link}(M_2, C)}\, H(C) \end{cases}$$

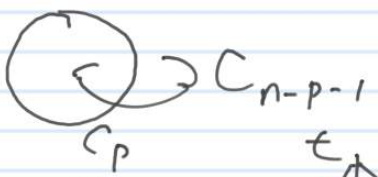
* linking

3d



← $\mathrm{Link}(C_1^{(1)}, C_1^{(2)}) = 1$

n-dim

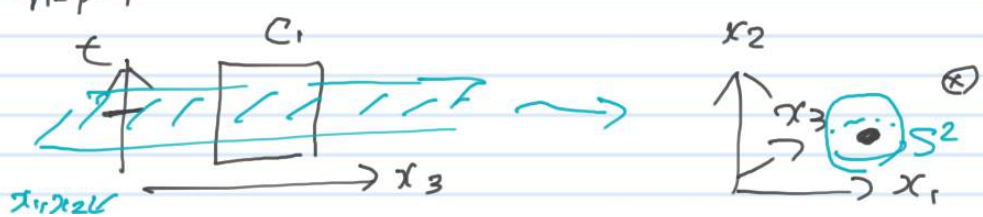


← $\mathrm{Link}(C_p, C_{n-p-1}) = 1$

4-dim

$C_1 = C$.

$C_2 = S^2$



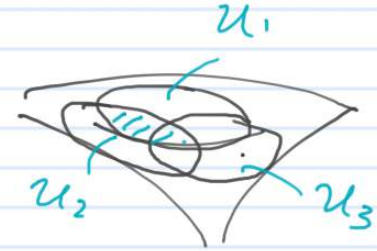
- $\mathbb{Z}_N^{(1)}$ - symmetry for $SU(N)$ Yang-Mills theory.

  $\rightarrow$ We have to construct codim-2 topological operator.

Mathematical data of $SU(N)$ YM theory

$\rightarrow$ a : $SU(N)$ gauge field $\Rightarrow f = da + i a \wedge a$.

$\hookrightarrow$ $M_4 \Rightarrow \{\mathcal{U}_\alpha\}$ open cover



On each $\mathcal{U}_\alpha$,

a_α : $\mathrm{Lie}(SU(N))$-valued 1-form field.

On each $\mathcal{U}_{\alpha\beta} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta$,

$g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \longrightarrow SU(N)$ (transition function

and

$$a_\beta = g_{\alpha\beta}^{-1} a_\alpha g_{\alpha\beta} - i g_{\alpha\beta}^{-1} d g_{\alpha\beta}.$$

On each $\mathcal{U}_{\alpha\beta\gamma} = \mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$,

$$g_{\beta\gamma} g_{\alpha\beta} = g_{\alpha\gamma}. \quad (\Longleftrightarrow g_{\alpha\gamma} g_{\beta\gamma} g_{\alpha\beta} = 1)$$

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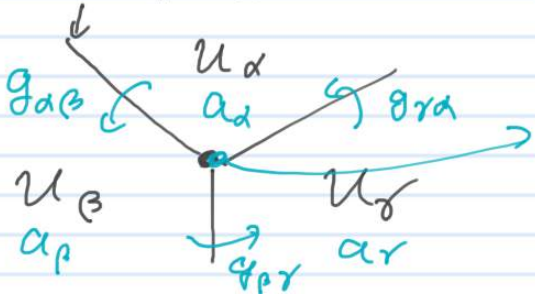
$\mathcal{U}_\alpha \cap \mathcal{U}_\beta$ shrinks to a line

$\Longleftrightarrow$

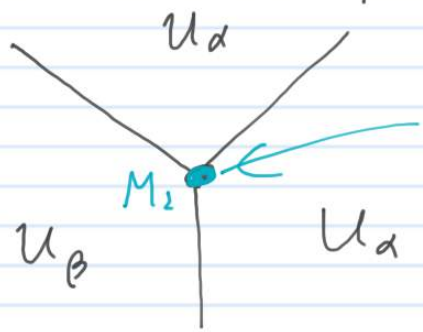


$$g_{\gamma\alpha} g_{\beta\gamma} g_{\alpha\beta} = 1.$$

* If all the matter fields are in the adjoint reprensation, then we can relax the cocycle condition $g_{\gamma\alpha} g_{\beta\gamma} g_{\alpha\beta} = 1$ to

$$g_{\gamma\alpha} g_{\beta\gamma} g_{\alpha\beta} = e^{\frac{2\pi i}{N} n_{\alpha\beta\gamma}} \in \underbrace{\mathbb{Z}_N}_{\text{center of } SU(N)}.$$

. Codim-2 $\mathbb{Z}_N$ operator $U_n(M_2)$ (Gukov-Witten surface operator)



cocycle condition

$$g_{\gamma\alpha} g_{\beta\gamma} g_{\alpha\beta} = e^{\frac{2\pi i}{N} n}$$

- $U_n(M_2)$ is a topological operator
- $$\langle U_n(M_2)\, W_\square(C) \rangle = e^{\frac{2\pi i}{N} n \cdot \mathrm{Link}(M_2, C)} \langle W_\square(C) \rangle$$

$\Longleftrightarrow$ $\mathbb{Z}_N$ transformation acting on the Wilson loop.

$\rightsquigarrow$ We gauge this $\mathbb{Z}_N^{(1)}$ and detect the mixed 't Hooft anomaly @ $\theta = \pi$.

- <u>Gauging 1-form symmetry</u>

cf. Gauging of $U(1)^{(0)}$ symmetry of 2π-periodic scalar:

$$\mathcal{L} = \frac{1}{2e^2} d\phi \wedge * d\phi \quad \left(= \frac{1}{2e^2} |d\phi|^2\right) \quad (\phi \sim \phi + 2\pi).$$

$\Phi \sim v e^{i\phi}$

This has a $U(1)$ symmetry $\phi \to \phi + \alpha$.
Noether current

$$j = \frac{1}{e^2} * d\phi$$

We want to introduce the $U(1)$ gauge field A in a minimal way

$$\mathcal{L} + \boxed{A \wedge j} + \cdots = \frac{1}{2e^2} |d\phi + A|^2.$$

$$= 2A \wedge \frac{1}{2e^2} * d\phi$$

$\mathcal{O}(A^2)$-term to achieve the local invariance.

We can do the same for $U(1)$ Maxwell theory

$$\mathcal{L} = \frac{1}{2e^2} da \wedge * da.$$

$U(1)_E^{(1)}$ is generated by

$$j_E = \frac{1}{e^2} * da.$$

We introduce a 2-form $U(1)$ gauge field B_E in a way that

$$\mathcal{L} + B_E \wedge j_E \left(+ \cancel{\frac{1}{2e^2} B_E \wedge * B_E}\right) = \frac{1}{2e^2} |da + B_E|^2.$$

The action is invariant under the local $U(1)$ 1-form transformation:

$$a \to a + \lambda, \qquad B_E \to B_E - d\lambda.$$

$\rightsquigarrow$ $U(1)_E^{(1)}$ is "gauged" by this operation.

- Gauging of $\mathbb{Z}_N^{(1)}$ for $SU(N)$ YM
(in continuum formulation (cf) Kapustin, Seiberg 2014).

Idea : $W \sim \mathrm{tr}(Pe^{i\oint a}) \xrightarrow{\mathbb{Z}_N^{(1)}} e^{\frac{2\pi i}{N}} W(C)$ ← global $\mathbb{Z}_N^{(1)}$-sym.

$\Updownarrow$ Gauging.

$a \to a'' + \lambda''$

We prepare 2-form gauge field B ⇒ we cook up "1-form gauge inv." field strength $f + B$

We pay attention to the fact. $U(N) = \dfrac{SU(N) \times U(1)}{\mathbb{Z}_N}$.

$(U(N) \ni U \Leftrightarrow U^\dagger U = 1)$

Then, we decompose $U = \underset{\in U(1)}{e^{i\alpha}} \underset{\in SU(N)}{V}$, but this decomp. is ambiguous

$\left(\because U = \underbrace{e^{i(\alpha + \frac{2\pi}{N})}}_{\in U(1)} \times \underbrace{(e^{-\frac{2\pi i}{N}} V)}_{\in SU(N)}\right)$ is also a good decomp.

If we consider $U(N)$ gauge theory instead of $SU(N)$,
1-form sym. is now $U(1)^{(1)}$ instead of $\mathbb{Z}_N \subset U(1)$.

So, by embedding $SU(N)$ YM $\Longrightarrow$ $U(N)$ YM,
then $\mathbb{Z}_N^{(1)}$ is embedded into $U(1)^{(1)}$.
we already know how to gauge this.

<u>Caveats of this idea</u>

$B_{U(1)}$: 2-form gauge field $\rightsquigarrow \exp(i\int_{M_2} B) \in U(1)$

- $\mathbb{Z}_N^{(1)}$ & $U(1)^{(1)}$ fluxes are different.
  more restrictive.

If we prepare $\underline{\underline{B}}$-field for $\mathbb{Z}_N$, $\exp(i\int B) \in \mathbb{Z}_N$.

$$\left(\begin{array}{l} \cdot\ H^2(M_4, U(1)) \ni B_{U(1)} \\ \quad H^2(M_4, \mathbb{Z}_N) \ni B \end{array} \right).$$

<u>Concrete construction</u>

a : $SU(N)$-gauge field.

We realize the $\mathbb{Z}_N$-valued 2-form gauge field
by preparing $U(1)$ 2-form gauge field B & $U(1)$ 1-form gauge field C,
and impose $\boxed{NB = dC}$

Flux quantization of B : M_2 : closed 2-submanifold.

$$\mathcal{O}_{M_2} = \exp(i \int_{M_2} B) \in U(1)$$

We take N-th power:

$$\mathcal{O}_{M_2}^N = \exp(i \int_{M_2} NB) \overset{NB = dC}{=} \exp(i \underbrace{\int_{M_2} dC}_{\in 2\pi \mathbb{Z}}) = 1.$$

$U(1)$ 1-form gauge field

(Dirac quantization)

Therefore,

$$\exp(i \int_{M_2} B) = e^{\frac{2\pi i}{N} n} \in \mathbb{Z}_N.$$

Now we embed $SU(N)$-gauge field a into a $U(N)$ gauge field:

$$\underbrace{\tilde{a}}_{\substack{U(N) \\ \sim SU(N)\times U(1)}} \text{"}= \underbrace{a}_{SU(N)} + \frac{1}{N} \underbrace{C}_{U(1)} \mathbb{1}_N\text{"}. \quad -(*)$$

"$N \times N$ identity matrix.

More precisely (or, more in a gauge-inv. way), we can state that we consider $U(N)$-gauge field $\tilde{a}$, and impose the condition

$$NB = dC = \mathrm{tr}(d\tilde{a} + i\tilde{a} \wedge \tilde{a})$$

one can easily check this equality from $(*)$

- We next check the 1-form gauge transformation:

$$B \to B + d\lambda. \qquad C \to C + N\lambda$$

This implies that

$$\tilde{a} \to \tilde{a} + \lambda \mathbb{1}_N.$$

So, we can construct the 1-form gauge inv field strength as

$$[\underbrace{(d\tilde{a} + i\tilde{a}\wedge\tilde{a})}_{=\tilde{f}} - B\,\mathbb{1}_N].$$

Therefore, we obtain the $\mathbb{Z}_N^{(1)}$-gauged Lagrangian

$$\mathcal{L}_{\text{ungauged}} = \frac{1}{g^2}\,\mathrm{tr}(f\wedge *f) + i\theta\frac{1}{8\pi^2}\,\mathrm{tr}(f\wedge f)$$

$$\Longrightarrow \quad \mathcal{L}_{\text{gauged}} = \frac{1}{g^2}\,\mathrm{tr}\big((\tilde{f}-B)\wedge *(\tilde{f}-B)\big) + i\frac{\theta}{8\pi^2}\,\mathrm{tr}\big((\tilde{f}-B)^2\big).$$

In the language of the partition function,

$$\boxed{Z_\theta[B] = \int \mathcal{D}\tilde{a}\Big|_{NB=\mathrm{tr}(\tilde{f})} \exp\left(-\int \mathcal{L}_{\text{gauged}}[\tilde{a}, B]\right).}$$

(Let's check $Z_\theta[B=0]$ is the usual YM partition function.

$$B=0 \iff \underset{\substack{\| \\ d(\mathrm{tr}(\tilde{a}))}}{\mathrm{tr}(\tilde{f})} = 0$$

Up to $U(1)^{(0)}$-gauge trans., $\mathrm{tr}(\tilde{a}) = 0. \iff \underline{\tilde{a} : SU(N)\text{ gauge field}}$)

From now on, we check the $\mathbb{Z}_N^{(1)} \times CP$ mixed anomaly @ $\theta = \pi$.

$\Rightarrow$ We have to look at the property of the topological charge. (van Baal '82)

$$Q_{top}[B] = \frac{1}{8\pi^2}\int \mathrm{tr}\left((\tilde{f}-B)\wedge(\tilde{f}-B)\right)$$

$$= \frac{1}{8\pi^2}\int \mathrm{tr}\left(\tilde{f}\wedge\tilde{f} - 2\tilde{f}\wedge B + B\wedge B\right)$$

$$= \frac{1}{8\pi^2}\int \mathrm{tr}(\tilde{f}\wedge\tilde{f}) - \frac{1}{8\pi^2}\int 2\underbrace{\mathrm{tr}(\tilde{f})}_{=NB}\wedge B + \frac{1}{8\pi^2}\int \overset{\mathrm{tr}(\mathbb{1}_N)}{N} B\wedge B$$

$$= \underbrace{\frac{1}{8\pi^2}\int \mathrm{tr}(\tilde{f}\wedge\tilde{f})}_{ch_2(U(N)) \in \mathbb{Z}} - \underbrace{\frac{N}{8\pi^2}\int B\wedge B}_{= \frac{1}{N}\underbrace{\frac{1}{8\pi^2}\int dC\wedge dC}_{\in \mathbb{Z}} \in \frac{1}{N}\mathbb{Z}}.$$

$(NB = dC)$ ← U(1) gauge field

Therefore,

$$Q_{top} \overset{\mathrm{mod}\,1}{=} \underbrace{-\frac{N}{8\pi^2}\int B\wedge B}_{} \in \frac{1}{N}\mathbb{Z}.$$

The fractionalized part is completely specified by the background B-field.

This means that

$$Z_{\theta+2\pi}[B=0] = Z_\theta[B=0] \rightsquigarrow Z_{\theta+2\pi}[B] = \underbrace{\exp\left(i\frac{N}{4\pi}\int B\wedge B\right)}_{} Z_\theta[B]$$

Under the CP transformation, $\theta \to -\theta$.

Before gauging $\mathbb{Z}_N^{(1)}$, $Z_{\theta=\pi}[B=0] \xrightarrow{CP} Z_{\theta=-\pi}[B=0] = Z_{\theta=+\pi}[B]$

CP-symmetric @ $\theta = \pi$.

After gauging $\mathbb{Z}_N^{(1)}$,

$$Z_{\theta=\pi}[B] \xrightarrow{CP} Z_{\theta=-\pi}[B] = \underbrace{\exp\left(-i\frac{N}{4\pi}\int B\wedge B\right)}_{\text{possible anomaly.}} Z_{\theta=+\pi}[B]$$

$\left(\begin{array}{l} N:\text{even} \to \text{genuine anomaly} \\ N:\text{odd} \to \text{no anomaly due to a local counterterm} \end{array}\right)$

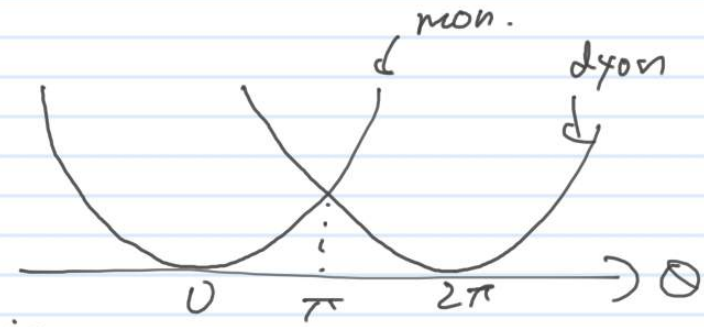
We thus found that $\mathbb{Z}_N^{(1)} \times CP$ is 't Hooft anomalous.

$\overset{\text{Anomaly matching}}{\Longrightarrow}$

- ~~CP~~ ($\leftarrow$ large-N, monopole condensation scenario)
- ~~$\mathbb{Z}_N^{(1)}$~~ ($\leftarrow$ Higgs, Oblique confinement (Honda, Tanizaki '20))
- Coulomb phase.

Reinterpretation from monopole cond.

The transition from mon/dyon cond. could be a smooth crossover.



However, now this is impossible.

$$Z_{\theta\approx 0}[B] = 1$$ ← We take this as a normalization

Then

$$Z_{\theta\approx 2\pi}[B] = \exp\left(i\frac{N}{4\pi}\int B\wedge B\right).$$ ← This is a consequence of anomaly relation.

$\Rightarrow$ $\theta=0$ & $\theta=2\pi$ vacua must be different as Symmetry — Protected Topological (SPT) phases with $\mathbb{Z}_N^{(1)}$.

Thus, we need to have a Quantum Phase transition to go from one to the another.

$\theta = \pi - 0$

mon. cond.

$S_{EFT} = 0$

$\theta = \pi + 0$

dyon cond.

$S_{EFT} = \frac{N}{4\pi} B \wedge B$

This is not gauge-inv. with boundary.

We need some local degrees of freedom to cancel the anomaly inflow.

$U(1)_{-N}$ Chern-Simons

(boundary deconfinement)